Lecture 4, Sep 22, 2021

Computing Projections

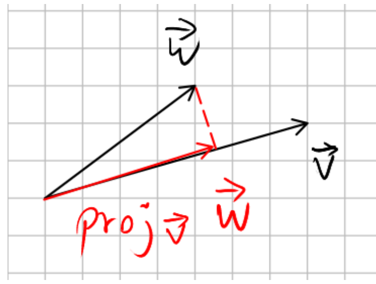

Figure 1: projection

- $\vec{u} = \text{proj}_{\vec{v}}\,\vec{w}$
- $\vec{u}$ has certain properties:
  1. $\vec{u} \parallel \vec{v} \iff \vec{u} = c\vec{v}$
  2. $(\vec{w} - \vec{u}) \perp \vec{v} \iff (\vec{w} - \vec{u}) \cdot \vec{v} = 0$
- Combine the equations to find c: $(\vec{w} - c\vec{v}) \cdot \vec{v} = \vec{w} \cdot \vec{v} - c\|\vec{v}\|^2 = 0 \implies c = \dfrac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}$
- Therefore $\vec{u} = \text{proj}_{\vec{v}}\,\vec{w} = \dfrac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$ or $\dfrac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}$
- Alternatively $\dfrac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \dfrac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|}\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|}\hat{v}$

Cross Products (aka Vector Products)

- Given $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \vec{u} = \vec{v} \times \vec{w} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ -v_1w_3 + v_3w_1 \\ v_1w_2 - v_2w_1 \end{bmatrix}$
- The result of the cross product $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, and has magnitude equal to the area of the parallelogram created by $\vec{v}$ and $\vec{w}$
- Properties of the cross product:
  1. Distributive: $\vec{v} \times (\vec{w} + \vec{z}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{z}$
  2. Anti-commutative: $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$
  3. Associative with scalars: $c\vec{v} \times \vec{w} = \vec{v} \times c\vec{w} = c(\vec{v} \times \vec{w})$
  4. $\vec{v} \times \vec{0} = \vec{0} \times \vec{v} = 0$
  5. **Not** associative with itself: $(\vec{v} \times \vec{w}) \times \vec{z} \neq \vec{v} \times (\vec{w} \times \vec{z})$
- The direction of $\vec{u}$ follows the right-hand rule: index finger in the direction of $\vec{v}$, middle finger in the direction of $\vec{w}$, then the thumb will be pointing in the direction of $\vec{u} = \vec{v} \times \vec{w}$
  - Alternatively start at $\vec{v}$ and curl towards $\vec{w}$
- Lagrange identity says that $\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2\|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2$
  - Use direct proof approach and use the definition of the cross product to brute force the proof
  - Visualization: Suppose θ is the angle between $\vec{w}$ and $\vec{v}$; then $\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2\|\vec{w}\|^2 - (\|\vec{v}\|\|\vec{w}\|\cos\theta)^2 = \|\vec{v}\|^2\|\vec{w}\|^2(1 - \cos^2\theta) = \|\vec{v}\|^2\|\vec{w}\|^2\sin^2\theta$
  - Therefore $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$ (absolute value not needed because $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$)
    * Similar to the dot product which is $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$
  - This corresponds to the area of the parallelogram defined by $\vec{v}$ and $\vec{w}$:
    *
    * The base length is $\|\vec{w}\|$, and the height is $\|\vec{v}\|\sin\theta$, so the magnitude of the cross product is the area of this parallelogram
- The dot product is a measure of how much the vectors are parallel; the cross product is a measure of

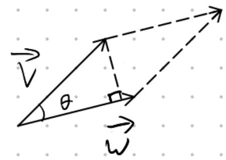

Figure 2: area of paralleogram

how much the vectors are orthogonal