Lecture 3, Sep 20, 2021

Properties of the Dot Product

- Dot products are **distributive**, **commutative**, and **associative** (with scalars: $c(\vec{v}\cdot\vec{w}) = c\vec{v}\cdot\vec{w} = \vec{v}\cdot c\vec{w}$)
- The dot product of a vector with itself is the square of its length: $\vec{v}\cdot\vec{v} = v_1^2 + v_2^2 + v_3^2 = \|\vec{v}\|^2$

Geometric Meaning of the Dot Product

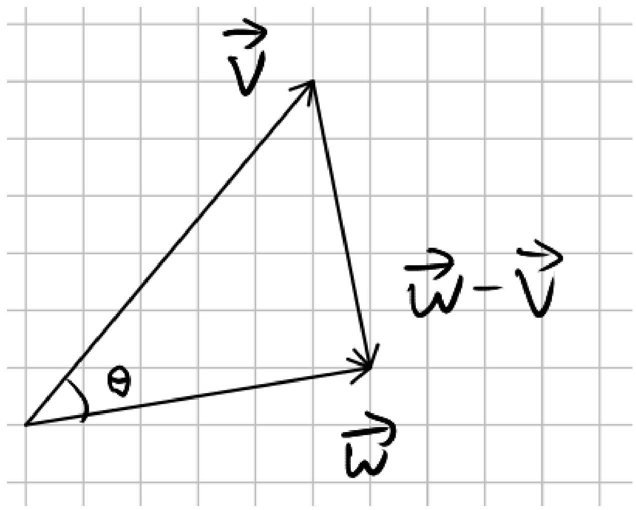

Figure 1: img

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- $\|\vec{w}-\vec{v}\|^2 = (\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v}) = \vec{w}\cdot\vec{w} - 2\vec{w}\cdot\vec{v} + \vec{v}\cdot\vec{v} = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\vec{w}\cdot\vec{v}$
  - Apply the cosine law: $\|\vec{w}-\vec{v}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{w}\|\|\vec{v}\|\cos\theta$
  - Therefore: $\vec{w}\cdot\vec{v} = \|\vec{w}\|\|\vec{v}\|\cos\theta \implies \cos\theta = \dfrac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|\|\vec{v}\|}$
- Since cosine has range $[-1, 1]$, this means the dot product can be negative; when the dot product is negative $\theta < \dfrac{\pi}{2}$ so the two vectors are in the same direction
- When $\vec{v}\cdot\vec{w} = 0$, then $\cos\theta = 0 \implies \theta = \dfrac{\pi}{2}$ so $\vec{v}$ and $\vec{w}$ are orthogonal

Two Important Inequalities

1. *Triangle Inequality*: $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

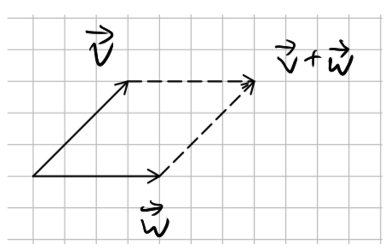

Figure 2: img

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- $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}+2\vec{v}\cdot\vec{w} = \|\vec{v}\|^2+\|\vec{w}\|^2+2\|\vec{v}\|\|\vec{w}\|\cos\theta$
- Replace $\cos\theta$ with its least upper bound 1: $\|\vec{v}+\vec{w}\|^2 \leq \|\vec{v}\|^2+\|\vec{w}\|^2+2\|\vec{v}\|\|\vec{w}\| = (\|\vec{v}\|+\|\vec{w}\|)^2$
- Take the square root: $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\|+\|\vec{w}\|$

2. *Cauchy-Schwartz-Bunyakovsky Inequality*: $|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$

Projections

- Key concept: **Orthogonality**: Two vectors are orthogonal iff $\vec{v}\cdot\vec{w}=0$
- To get the components/coordinates of a vector we project it onto $\hat{\imath},\hat{\jmath},\hat{k}$, etc (standard projections)
- The projection of $\vec{w}$ onto $\vec{v}$, $\text{proj}_{\vec{v}}\,\vec{w}$ is a vector parallel to $\vec{v}$ that is as close as $\vec{w}$ as possible:
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