Lecture 21, Dec 6, 2021

LU Decomposition

- Let A be an invertible square $n \times n$ matrix
- We can write A in a factored form as $A = LU$, where L is a lower-triangular matrix (zeros above the diagonal), and U is an upper-triangular matrix (zeros below the diagonal)
- To solve $A\vec{x} = \vec{b}$ using LU decomposition:
  1. Rewrite $A\vec{x} = \vec{b} \implies LU\vec{x} = \vec{b}$, and let $\vec{y} = U\vec{x}$, then $LU\vec{x} = \vec{b} \implies L\vec{y} = \vec{b}$
  2. Solve for $\vec{y}$
  3. Substitute $\vec{y}$ in $U\vec{x} = \vec{y}$, and solve for $\vec{x}$
- We've separated the decomposition phase, which is computationally heavy, and the part where $\vec{b}$ is brought in, which is computationally light
  - LU decomposition is better than computing the inverse, because if A is sparse computing the inverse might lead to a lot of numerical errors
- Because L and U have special forms, solving $U\vec{x} = \vec{y}$ is much easier
- LU factorizations aren't unique, but to solve the problem we can use any one
- Example: $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$
  1. Factor $A = LU \implies \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$
  2. Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = U\vec{x}$
  3. $L\vec{y} = \vec{b} \implies \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \implies \vec{y} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$
     - By forward substitution $\begin{cases} 2y_1 = 2 & \implies y_1 = 1 \\ -3y_1 + y_2 = 2 & \implies y_2 = 5 \\ 4y_1 - 3y_2 + 7y_3 = 3 & \implies y_3 = 2 \end{cases}$
  4. $U\vec{x} = \vec{y} \implies \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \implies \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$
     - By back substitution $\begin{cases} x_3 = 2 & \implies x_3 = 2 \\ x_2 + 3x_3 = 5 & \implies x_2 = -1 \\ x_1 + 3x_2 + x_3 = 1 & \implies x_1 = 2 \end{cases}$
- To find LU:
  - During elimination we have $E_n \cdots E_1 A = U$, so $L = E_1^{-1} \cdots E_n^{-1}$
  - As long as we don't do any back substitution, $E_1^{-1} \cdots E_n^{-1}$ is going to be lower-triangular