Lecture 20, Nov 29, 2021

Boundary Value Problems

- A boundary value problem involves a system described by a differential equation, but unlike an IVP, the known values are no longer at 0
- In a BVP the independent variable is typically not time but some kind of spacial variable (hence boundaries)
- Example: Beam deflection problem; in this case the known points are the supports which have a deflection of 0
- To solve BVPs numerically, begin by partitioning the interval into n evenly spaced subintervals, and set the step size as $\Delta x = \dfrac{b-a}{n}$
- Approximate derivatives using finite differences (secant line slopes)
  - The forward difference $\Delta_F f(x) = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$
  - The backward difference $\Delta_B f(x) = \dfrac{f(x) - f(x-\Delta x)}{\Delta x}$
  - The central difference $\Delta_C = \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$
  - We can use different combinations of these finite differences to estimate higher order derivatives
- $$\begin{aligned} f''(x) &= \Delta_B[\Delta_F f(x)] \\ &= \frac{\Delta_F f(x) - \Delta_F f(x-\Delta x)}{\Delta x} \\ &= \frac{1}{\Delta x}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x} - \frac{f(x)-f(x-\Delta x)}{\Delta x}\right] \\ &= \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} \end{aligned}$$
- Example: Given $y'' + 2y' + y = 0$ and $y(0) = 0$, $y(1) = 1$, find y that satisfies this for $x \in [0,1]$
  - Approximate $y'(x) \approx \Delta_C y(x)$ and $y''(x) \approx \Delta_B \Delta_F y(x)$
  - Substitute this back into the equation: $\dfrac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} + 2\dfrac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x} + y(x) = 0$
  - Simplify: $\left(\dfrac{1}{(\Delta x)^2} + \dfrac{2}{2\Delta x}\right) y(x+\Delta x) + \left(\dfrac{-2}{(\Delta x)^2} + 1\right) + \left(\dfrac{1}{(\Delta x)^2} - \dfrac{2}{2\Delta x}\right) y(x-\Delta x) = 0$
  - This is now a difference equation, with no more derivatives but instead values of y at discrete points
  - Divide the interval $[0,1]$ into $n = 5 \implies \Delta x = 0.2$ subintervals
  - Now we evaluate the difference equation at each of the interior grid points (open interval $(0,1)$), which will get us an algebraic equation:
    - $$\begin{cases} 30y(0.4) - 49y(0.2) + 20y(0) = 0 \\ 30y(0.6) - 49y(0.4) + 20y(0.2) = 0 \\ 30y(0.8) - 49y(0.6) + 20y(0.4) = 0 \\ 30y(1) - 49y(0.8) + 20y(0.6) = 0 \end{cases}$$
      - Note how the coefficients aren't changing since they only depend on step size
  - Now we can sub in $y(1)$ and $y(0)$ into the equations above
  - The system of y values can now be formulated in the form of $A\vec{x} = \vec{b}$, which allows it to be solved