Lecture 19, Nov 24, 2021

Systems of Linear Differential Equations

- Consider the IVP: $\begin{cases} x'(t) = ax(t) + by(t) \\ y'(t) = cx(t) + dy(t) \end{cases}$
  - This is a coupled system so we can't simply use the method in the last lecture to the two equations separately
  - We can write this using a matrix: Let $Z = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, then $Z' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$, so the system can now be represented as $Z' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} Z = AZ$
  - To capture the initial conditions, let $Z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$
  - This is referred to as a *state-space* description of the system and only applies to systems described by linear differential equations with constant coefficients
    - Z is the *state* of the system
  - For these systems there are analytical solutions
- Euler's method can be applied to such a system in the same way
  - Let $t_{n+1} = t_n + \Delta t$
  - $Z_{n+1} = Z_n + \Delta t Z'_n = Z_n + \Delta t A Z_n$
- Improved Euler's method:
  - $Z^*_{n+1} = Z_n + \Delta t A Z_n$ is the Euler update
  - $Z_{n+1} = Z_n + \dfrac{\Delta t}{2}(AZ_n + AZ^*_{n+1})$ is the improved update, where the average of the rate of change at Z_n and Z^*_{n+1} is averaged

Higher Order Systems

- Higher order systems of linear differential equations can be converted into systems of first order linear ODEs, by making the derivatives a part of the state
- Consider a general second-order differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} = y''(t) = f(t, y(t), y'(t))$, and for initial values $y(0) = y_0$ and $y'(0) = y'_0$
- Let $Z = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \implies Z' = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$, and $Z_0 = \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}$ and if the system is linear, we can express this in the same way using a matrix as a system of first order equations
- Example: $y''(t) = -y(t)$, $y(0) = 1$, $y'(0) = 0$
  - Set up $Z' = AZ$, where $Z = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \implies Z' = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$
  - $\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \implies A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
  - Now Euler's method can be used in exactly the same way