Lecture 15, Nov 3, 2021

Inverses by Gaussian Elimination

- To find the inverse using Gaussian elimination, we introduce elementary matrices for GE operations:
  - Swapping rows: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$
  - Multiplying a row by a scalar: $\begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ c & d \end{bmatrix}$
  - Add or subtract multiples of other rows: $\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + ec & b + ed \\ c & d \end{bmatrix}$
- Notice that the elementary matrices we obtained for these operations can also be obtained by applying the same operation to the identity!
  - Swapping rows, multiplying rows, or adding or subtracting multiples of rows in the identity gets us the elementary matrix we need
  - Suppose the elementary matrix is E and applies the operation we need; then because $EI = E$ and we know what E does to a matrix, we can calculate EI to get E
- Elementary matrices are one operation from the identity, so we can return back to the identity by applying its inverse (elementary matrices are invertible)
- Now apply GE to $[A|I]$, which corresponds to solving $AX = I$
  - As we do this we'll get $E_n E_{n-1} \cdots E_2 E_1 A$
  - Since applying GE to an invertible matrix produces I, we have $E_n E_{n-1} \cdots E_2 E_1 A = I$, therefore $E_n E_{n-1} \cdots E_2 E_1 = A^{-1}$
  - Since we're also applying the same operations to the right side, the right side becomes $E_n E_{n-1} \cdots E_2 E_1 I = E_n E_{n-1} \cdots E_2 E_1$, which is what we want
  - Therefore we can apply GE to $[A|I]$, and whatever becomes of I when A is reduced to I is A^{-1}
- Then $A = (E_n E_{n-1} \cdots E_2 E_1)^{-1}$ since $(A^{-1})^{-1} = A$; thus $A = E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1} E_n^{-1}$
  - This leads to a property that any invertible matrix A can be written as a product of elementary matrices