Lecture 14, Nov 1, 2021

Nonsquare Matrices

- When $m > n$, there are more equations than unknowns; $\operatorname{rank} A \leq n$; if $\operatorname{rank} A = n$ then there are no solutions or 1 solution
- When $m < n$, there are more unknowns than equations; $\operatorname{rank} A \leq m$; if $\operatorname{rank} A = m$, then there are always infinite solutions
- Full rank square matrices always have only 1 solution (full rank means $\operatorname{rank} A = \min\{m, n\}$)
- Any system that is not full rank can have either zero or infinite solutions

More About Inverses

- If we have the inverse of A for $A\vec{x} = \vec{b}$, then we can do $A^{-1}A\vec{x} = A^{-1}\vec{b} \implies I\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}$
- Definition: The square matrix A is invertible $\iff \exists B$ such that $AB = I$ and $BA = I$
- Inverses are unique: Suppose both B and C are inverses of A, then $BA = I = AC \implies BAC = CAC \implies B = C$, so all inverses are the same
- Properties of inverses:
  1. If A^{-1} exists, then when $[A|\vec{b}]$ is taken into rref, it becomes $[I|\vec{d}]$ where each variable is a leading variable
  2. If $A\vec{x} = \vec{0}$ has nontrivial solutions, then A does not have an inverse; i.e. matrices with a null space of more than the zero vector are not invertible
     - Proof: Assume A^{-1} exists, then $A^{-1}A\vec{x} = A^{-1}\vec{0} \implies I\vec{x} = \vec{0} \implies \vec{x} = \vec{0}$; this means that if A is invertible then $\vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$
  3. If matrices A and B are both invertible and of the same size, then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$
     - Proof: $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$, and $ABB^{-1}A^{-1} = A^{-1}IA = A^{-1}A = I$, so by the definition of the inverse $B^{-1}A^{-1}$ is the inverse of AB
  4. If A is invertible, then A^{-1} is also invertible, and $(A^{-1})^{-1} = A$
  5. Every invertible matrix A can be expressed as a product of elementary matrices