Lecture 23, Nov 2, 2021

Curvature Diagrams

- Like shear and moment diagrams, we can draw curvature diagrams
- Recall $\phi = \dfrac{M}{EI}$, so we can use this and the bending moment diagram to draw a curvature diagram (note EI is usually a constant, so the BMD and curvature diagrams are usually the same shape, just scaled by a constant; EI is not constant if the shape has a changing cross section or is not made of a uniform material)
  - The units of ϕ is rad/mm
  - Curvature can have jumps due to changing EI, but the moment diagram is always continuous
- Note that the y in $\sigma = \dfrac{My}{I}$ is not the same as the y in $\phi = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ (for small angles only); the first is internally in the member itself, the second is the deflection of the beam ($y(x)$ is the displaced shape of the beam)
  - If angles are not small, then $\phi = \dfrac{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}$
  - Angles are usually very small so $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is small and even more so when squared; the difference it causes is usually less than the uncertainties of the material constants so it's irrelevant
- Even though we know the curvature, we still don't know the displaced shape $y(x)$; to do this we need to double integrate, but this is a lot of work
  - Integrating ϕ gets θ, and integrating θ gets us $y(x)$
  - Example: Consider a beam with a wkN/m distributed load on top; the max bending moment is $\dfrac{wL^2}{8}$ can can be modelled by $M(x) = \dfrac{x(L-x)w}{2}$
    * The curvature is $\dfrac{x(L-x)w}{2EI} = \dfrac{wxL}{2EI} - \dfrac{wx^2}{2EI}$
    * $\theta = \displaystyle\int \phi \,\mathrm{d}x = \dfrac{wx^2L}{4EI} - \dfrac{wx^3}{6EI} + C_1$
    * $y = \displaystyle\int \theta \,\mathrm{d}x = \dfrac{wx^3L}{12EI} - \dfrac{wx^4}{24EI} + C_1x + C_2$
    * Using our initial conditions, $C_2 = 0$ and $C_1 = -\dfrac{wL^3}{24EI}$
    * To test this we can evaluate at $x = \dfrac{L}{2}$, since the structure is symmetric the slope should be 0 at this point
    * $y(x) = \dfrac{wx^3L}{12EI} - \dfrac{wx^4}{24EI} - \dfrac{wL^3x}{24EI}$
    * Note the signs might be weird
  - Even though it might be easy in this case, it is usually very hard; in this case we had a single continuous equation for the bending moment, but if we had point loads or other more complex loading types there will be lots of pieces in the moment equation
  - The moment equation is usually hard to obtain
- To get the shape more easily, we can use the two Moment Area Theorems

First Moment Area Theorem (θ)

- Since curvature $\phi = \dfrac{\mathrm{d}\theta}{\mathrm{d}x}$ the change in slope over the change in length, so by the fundamental theorem of calculus, the change in slope between two points $\Delta\theta_{AB}$ is the integral of the curvature
  - Note $\theta(x)$ is the slope of the beam at x
- $\Delta\theta_{AB} = \displaystyle\int_A^B \phi(x)\,\mathrm{d}x$ is the first Moment Area Theorem, which states that **the change in slope**

between any two sections of a deflected beam is equal to the area under the curvature diagram between those two sections

- Note that since this method only gives us the change, we need a point where the slope is known
- Signs are not automatic and must be determined using intuition

Second Moment Area Theorem (Δ)

- $\Delta = \int \theta \, \mathrm{d}x$ so we can do an integration-like process
- θ is the area under the curve between two regions plus a constant C_1
- $\Delta = \int (A + C_1) \, \mathrm{d}x$
- If we pretend that A is not a function of x, then we get $Ax + C_1x + C_2$
- But since A is a function of x, we have to multiply by $\bar{x}$ instead of x for $A\bar{x} + C_1x + C_2$, where $\bar{x}$ is the distance to the **centroid of the area under the curvature diagram**
- $C_1x + C_2$ is a line, which tells us that our answer is going to have a linear offset
- Geometrically this line is the tangent to the displaced shape at A
- Note δ_{BA} has the displacement at B, the tangent at A, and the area under the curvature diagram between A and B
- If the area under the curve is complicated we can break it up into pieces and have $\delta_{BA} = \sum_{i=1}^{n} A_i d_i$, where A_i is the area of each piece and d_i is the horizontal distance between B and the centroid of the piece i
- **For any two points A and B along the deflected beam, the tangential deivation of point B, δ_{BA}, is equal to the product of the area under $\phi(x)$ betweeen A and B, and the distance from the centroid of the diagram between A and B, to B (i.e. the first moment of area about point D)**

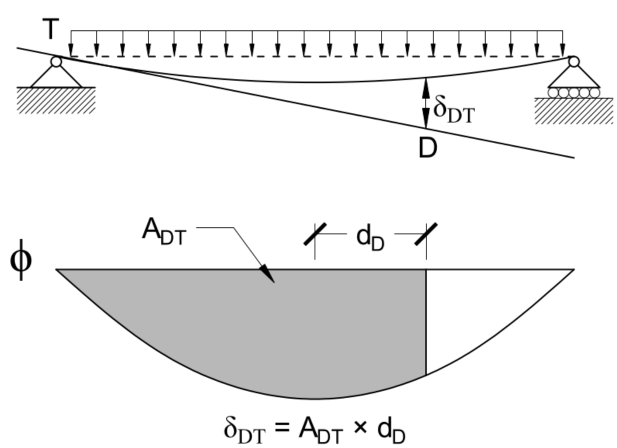

Figure 1: Summary of Moment Area Theorem 2; δ is the *tangential deviation*

Areas and Centroids of Common Shapes

- When using these two theorems, the areas and centroids of many different shapes are needed; these are available in the appendix
- If the cross sections are more complex but can be broken down into a number of common shapes, then we can still apply the two Moment Area Theorems and sum up the contributions from each piece

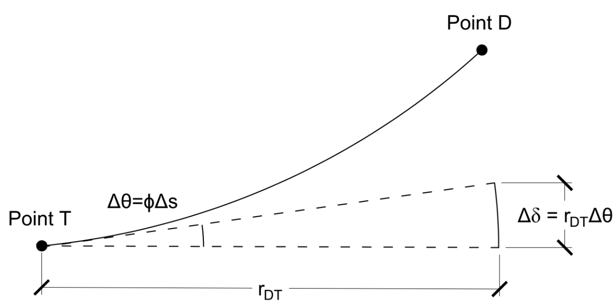

Figure 2: Diagram of the derivation of the Second Moment Area Theorem

- $$\begin{cases} \Delta\theta_{AB} = \sum i = 1^n \left[\int \phi(x)\,\mathrm{d}x \right]_i \\ \delta_{DT} = \sum_{i=1}^{n} \left[\bar{x} \int \phi(x)\,\mathrm{d}x \right]_i \end{cases}$$