Lecture 19, Oct 25, 2021

Damped Free Vibrations

- Oscillating systems will eventually stop due to loss of energy by *damping*, which could be engineered or an inherent system property
- Damping is quantified by the damping ratio β, the ratio between the provided damping properties of the system and the minimum amount of damping to prevent it from oscillating completely
  - *Critical damping* is the amount of damping that can make the system stop oscillating the fastest
  - More damping (overdamped) will make the system reach equilibrium slower, and less damping (underdamped) will cause it to oscillate
- In civil structures β range from 0 to 0.05
- For a damped system, the differential equation becomes $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\beta\sqrt{mk}\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0$
- The solution is an exponentially decaying sinusoid with the form $x(t) = Ae^{-\beta\omega_n t}\sin(\omega_d t + \phi) + \Delta_0$, where:
  - A is the amplitude of oscillation
  - β is the damping ratio
  - ω_n is the natural frequency $\sqrt{\frac{k}{m}}$
  - ϕ is the phase delay
  - Δ_0 is where the system naturally settles due to gravity
  - ω_d is the *damped frequency*, related to ω_n by $\omega_d = \omega_n\sqrt{1-\beta^2}$
    - Since β is usually low in civil structures, $\omega_d \approx \omega_n$

Forced Oscillations

- In reality structures may be subjected to dynamic loading due to the movement of people etc; in the simplest case the load is sinusoidal with $F(t) = F_0\sin(\omega t)$
- Substituting the dynamic load into the equation: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\beta\sqrt{mk}\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\sin(\omega t)$
- The complete solution is outside the scope of this course but consists of the sum of two parts: the *transient* solution, which dominates when t is small, and the *steady-state* solution, which dominates when t is large and dictates the long-term behaviour of the system
- The steady state solution is more relevant to design and can be expressed as $x(t) = \mathrm{DAF}\cdot\frac{F_0}{k}\sin(\omega t + \phi) + \Delta_0$
  - DAF is the *dynamic amplification factor*, calculated as $\frac{1}{\sqrt{\left(1-\left(\frac{f}{f_n}\right)^2\right)^2 + \left(\frac{2\beta f}{f_n}\right)^2}}$, where f is the driving frequency and f_n is the natural frequency of the system
  - From the equation for the DAF we can see that the response of the system is strongly influenced by $\frac{f}{f_n}$, as it influences both the frequency and the amplitude of the response through the DAF

Resonance

- The DAF has the highest value when the driving frequency is approximately equal to the natural frequency, which leads to a very high increase in amplitude known as *resonance*
- Increasing the amount of damping increases β and reduces the DAF, especially the peak value at resonance
- The DAF is 1 when $\frac{f}{f_n}$ is 0, and gradually becomes 0 as the ratio $\frac{f}{f_n}$ becomes large, with a peak at $\frac{f}{f_n} = 1$

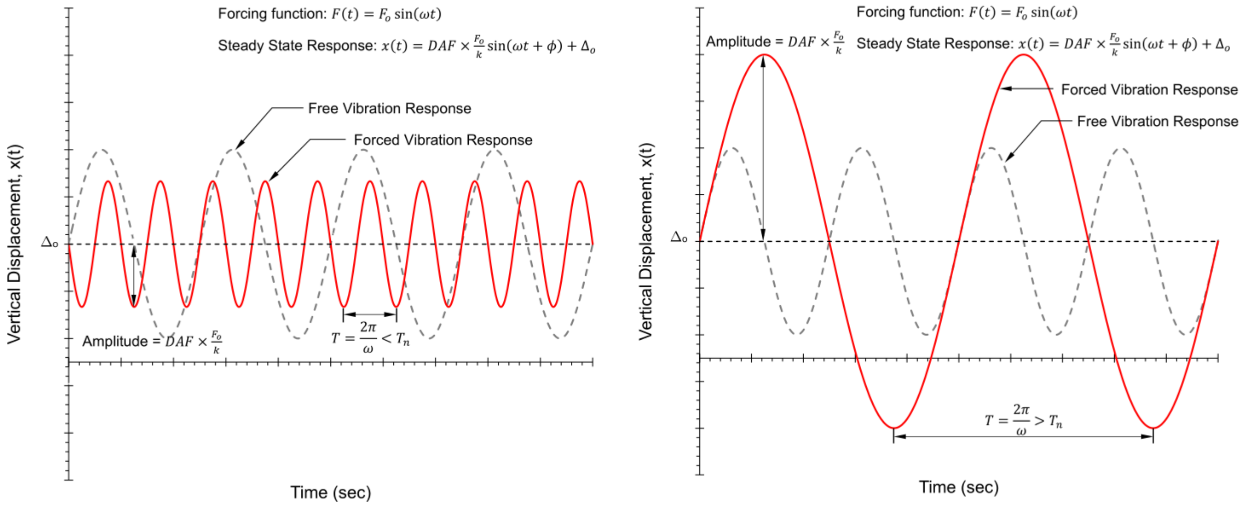

Figure 1: Response to an input load with $\dfrac{f}{f_n} = 2.5$ (left) and $\dfrac{f}{f_n} = 0.5$

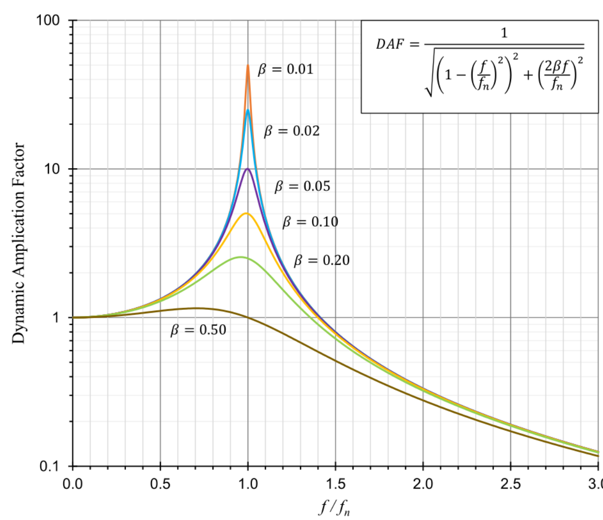

Figure 2: Influence of $\dfrac{f}{f_n}$ on the DAF for various values of β

Designing for Dynamic Effects

- To check for dynamic effects, we don't need to solve the whole system; we just need to check whether the maximum stress results in failure
- Consider a set of dynamic loads $w_{tot} = w_{sta} + w_0 \sin(\omega t)$, where w_{sta} is the stationary component of loading, which does not vary in time, such as the dead load of the structure and the weight of a standing crowd of people; w_0 and ω_0 is the dynamic load caused by, for example, a crow of people walking around
- When designing a pedestrian bridge the frequency of loading caused by walking is typically assumed to be 2Hz; therefore unless there is significant damping, having a natural frequency close to 2Hz can lead to large oscillations and possibly collapsing the structure
- The equivalent static load is then $w_{eq} = w_{sta} + \text{DAF} \cdot \omega_0$ as the maximum amplitude of oscillation is scaled by the DAF
- The DAF can be calculated once the damping β and natural frequency f_n are determined; using the equivalent static load we can then check whether the members can withstand the stresses
- For a point load at the midspan, the natural frequency can be estimated to be $f_n = \dfrac{15.76}{\sqrt{\Delta_0}}$; for a UDL at the midspan $f_n = \dfrac{17.76}{\sqrt{\Delta_0}}$, where Δ_0 is the midspan deflection under w_{sta} in mm, i.e. how much the midspan of the structure deforms with just the static load