Lecture 14, Oct 13, 2021

Euler Buckling

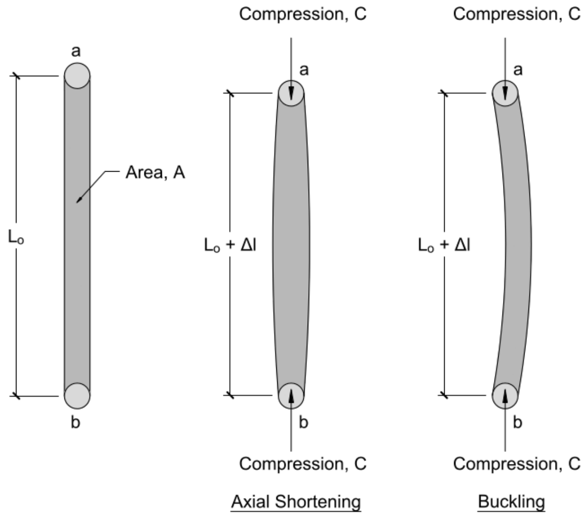

Figure 1: Two methods of failure for members under compressive strain

- Methods under compression can fail in 2 ways:
  1. *Crushing*: Typically occurs for short, stocky members; the member shortens; the force that causes crushing is the *squash load*
     - The squash load is simply $P_c = \sigma_c A$
  2. *Buckling*: Typically occurs for long, thin members; the member curves and folds
     - The load that causes this is the *Euler load* $P_e = \dfrac{\pi^2 EI}{L^2}$
- The original length is the longest, the buckled length is less long and the shortest length is the crushed length; since the buckled length is greater than the crushed length, it has less strain energy and so it is the lower energy state so nature prefers it
- The method in which a compressive member fails depends on the force required to cause crushing or buckling; whichever requires less force will be the method of failure

Derivation of Euler Load

- Fundamental assumptions:
  1. The material is homogeneous and linear elastic (constant Young's modulus E and second moment of area I)
  2. The ends are free to rotate and the top is free to move vertically (i.e. a roller at the top, a pin at the bottom)
  3. The member starts straight and the ends cannot move horizontally
- At the cut, the compressive force P and the vertical support are equal and in opposite directions and separated by a distance, so they form a couple; to resist it the beam carries an internal clockwise moment M
  - Since P and the vertical support force form a couple, $Py = M = EI\phi$
  - Recall that the curvature is the derivative of the slope, which is the second derivative of displacement; since the second derivative is negative in this case, we have $\phi = -\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$
  - The curvature is the negative of the second derivative; note when we have a positive slope (curvature) the beam is concave down so the second derivative is negative
    * Explained more in lecture 20

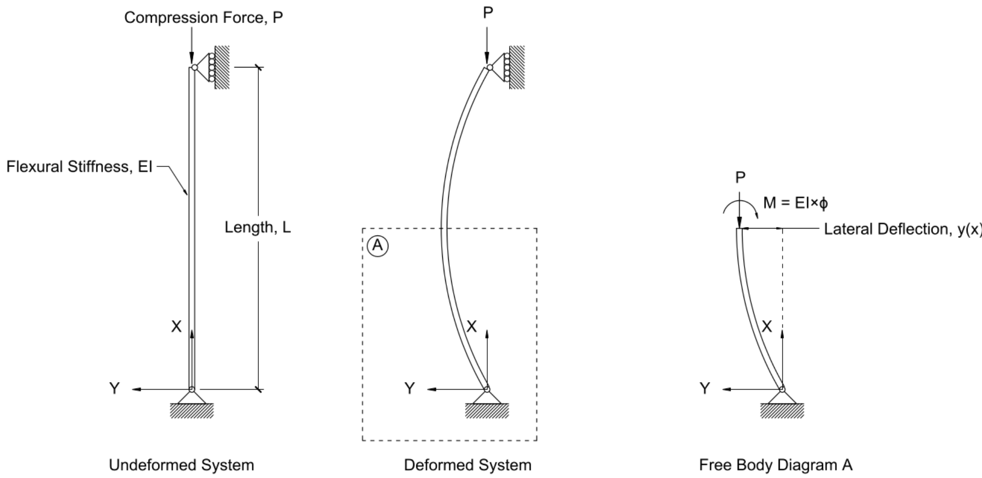

Figure 2: Derivation of the Euler load

- Combine the equations to get $\frac{P}{EI}y = -\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, which is a differential equation with solution in the form of $y = A\sin(\omega x + B)$; substituting this solution yields $\omega = \sqrt{\frac{P}{EI}}$
  - Since the ends are fixed, $y(0) = 0$ and $y(L) = 0$
    * Substituting $y(0) = 0 \implies A\sin(B) = 0 \implies B = 0$
    * And then $y(L) = 0 \implies A\sin(\omega L) = 0 \implies \omega L = n\pi \implies \omega = \frac{n\pi}{L}$ for $n \in \mathbb{Z}$
  - Substitute $\frac{n\pi}{L} = \omega = \sqrt{\frac{P}{EI}} \implies \frac{n^2\pi^2}{L^2} = \frac{P}{EI} \implies P = \frac{n^2\pi^2 EI}{L^2}$
  - To obtain the smallest nonzero value for P we take $n = 1$ to get $P_e = \frac{\pi^2 EI}{L^2}$

Higher Modes of Buckling

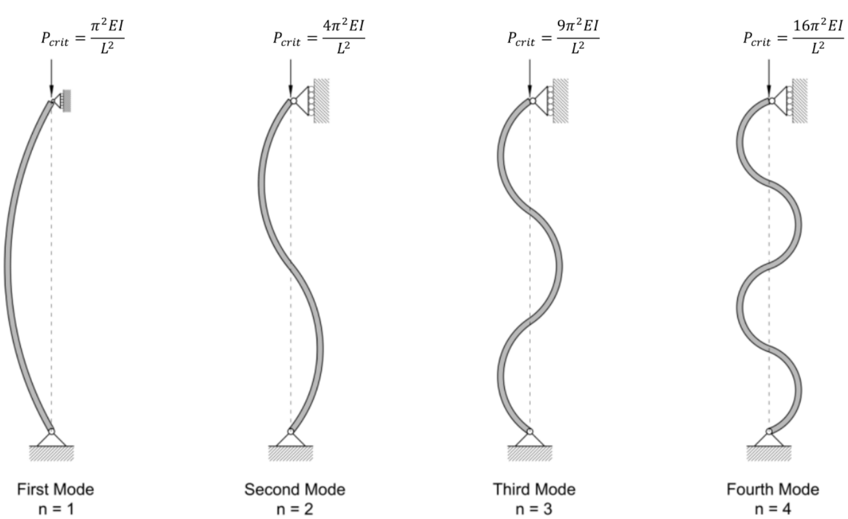

Figure 3: Higher modes of buckling

- The n term in $P = \frac{n^2\pi^2 EI}{L^2}$ and $\omega = \frac{n\pi}{L} \implies y(x) = A\sin\left(\frac{n\pi}{L}x\right)$ lets us “choose” the number of half-oscillations y goes through down the entire beam

- The higher modes of buckling do occur but are much more rare because the P force is much greater
- Oscillations in y correspond to more complex types of buckling, where instead of curving into 1 arc, the beam curves into a sine wave; this corresponds to higher modes of buckling

Buckling of Imperfect Members

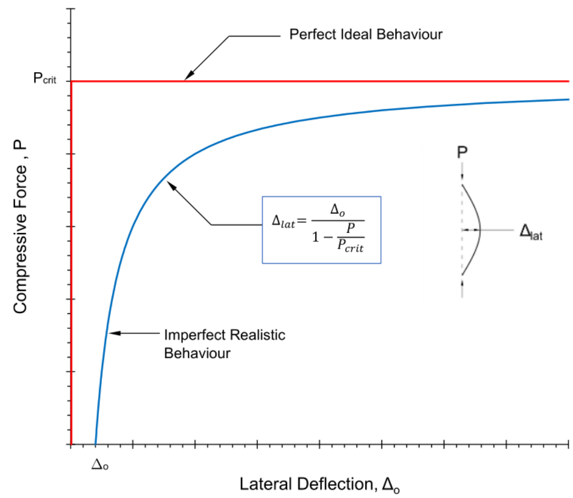

Figure 4: Ideal vs. realistic buckling curves

- Unlike tension, buckling is an unstable equilibrium; once a member starts to buckle it will continue to weaken and curve more until it fails
- Under ideal conditions members will stay straight until the Euler load is reached, at which point they suddenly buckle and the force stays constant
- In real life, as the angle of buckling gets larger, the force required gets smaller
- However, real members are imperfect and have some initial lateral deflection at their midpoint Δ_0
- Richard Southwell derived the equation $\Delta_{lat} = \dfrac{\Delta_0}{1 - \frac{P}{P_{crit}}}$ that relates lateral deflection Δ_{lat} to load P
  - The P_{crit} is the critical buckling load, and for members satisfying the Euler conditions (homogeneous, elastic, roller at one end and pin at the other) this is equal to the Euler load P_e